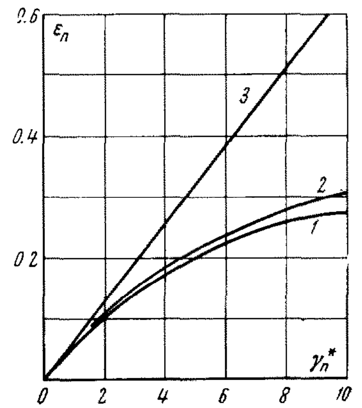

Fig. 1

$$\left[\frac{8}{5}\frac{3-6\nu+\nu^2}{(1+\nu)^2}\frac{h^2}{a^2}\gamma_n^2+\frac{48(1-\nu)^2}{1+\nu}\right]p_1^{*2}-$$
$$-\left[\frac{16}{5}\frac{3-2\nu}{1+\nu}\frac{h^2}{a^2}\gamma_n^2+48(1-\nu)\right]p_1^*+$$
$$+4\frac{h^2}{a^2}\gamma_n^2=0$$

where γ_n are zeros of the Bessel function $J_1(\gamma_n)=0$.

In this case the second boundary condition

$$\mathbf{e}_r\cdot(\mathbf{V}_1+\mathbf{V}_3)=0 \quad \text{for } r=a$$

is also satisfied.

In Fig. 1 the curve 1 represents the relationship $\varepsilon_n=\varepsilon_n(\gamma_n^*)$ for $\nu=0.3$ of the critical relative shortening of the plate radius

$$\varepsilon_n=1-\beta_n=\frac{1-\nu}{1+\nu}p_{1n}^*, \quad \gamma_n^*=\gamma_n^2\frac{h^2}{a^2}$$

The curve 2 corresponds to the exact solution of axisymmetric bifurcation of equilibrium of a circular cylinder compressed on the lateral surface by a uniform pressure. This result was obtained in [5].

The straight line 3 corresponds to the classical linear theory of buckling plates.

BIBLIOGRAPHY

1. Lur'e, A. I., Theory of elasticity for a semilinear material. PMM Vol. 32, №6, 1968.
2. Novozhilov, V. V., Fundamentals of Nonlinear Theory of Elasticity. M.-L., Gostekhizdat, 1948.
3. Naghdi, P. M., On the theory of thin elastic shells. Quart. J. Appl. Math. Vol. 14, №4, 1957.
4. Timoshenko, S. P. and Woinovski-Krieger, S., Plates and Shells. M., Fizmatgiz, 1963.
5. Sensenig, C. B., Instability of thick elastic solids. Commun. Pure and Appl. Math. Vol. 17, №4, 1964.

Translated by B. D.

ON THE CORRECTNESS OF CERTAIN PROBLEMS OF THE MEMBRANE THEORY OF SHELLS OF NEGATIVE CURVATURE

PMM Vol. 33, №4, 1969, pp. 676-687

V. A. SHACHNEV
(Moscow)

(Received July 5, 1968)

The determination of the state of stress and strain of a membrane shell of negative curvature reduces to the requirement of solving a system of hyperbolic-type equations. The boundary value problem for such a system does not always have a solution, and hence, such a problem is not generally correct. The following boundary value problem will be examined herein for the system of membrane theory equations in the case of shells of revolution of negative curvature. One static and one geometric tangential boundary condition [1] will be given on each edge of the shell, i.e. one stress resultant and one displacement in the plane tangent to the shell middle surface will be given at each point of the edge. In this case the problem separates into two: a static problem consisting of solving the equilibrium equations with the static boundary condition taken into account, and a geometric problem consisting of solving the equations for the displacements which govern small flexures of the shell, with the geometric boundary condition taken into account. If the directions of the displacements and stress resultants given on the shell edge are mutually orthogonal here, both problems will be conjugate. Because of the existence of alternative theorems in this case [1], the solvability of one problem defines the condition for solvability of the other. Each of these problems will be a boundary value problem for the appropriate hyperbolic system of equations, and the investigation of the correctness is made first for the static problem. It will be show here in addition that when just geometric tangential boundary conditions are given, but two on each edge of the shell, then the problem turns out to be correct.

The boundary value problem for hyperbolic equations has been examined in the literature, in particular for the equation or system of equations of string vibrations in [2–6]. It has been shown in the case of Dirichlet problem for the string equation that the boundary value problem cannot have a solution if the ratio of the sides of the rectangular domain in which the solution is defined is a rational number. The presence of an everywhere dense set of inadmissible domain dimensions results in the need to define the conditions under which the boundary value problem may be posed correctly. Some correctness conditions for the boundary value problem for the system of equations of string vibrations given in a square domain have been examined by Sobolev [4].

The boundary value problem in shell theory has been considered for some particular shells of negative curvature by Vlasov [7] and Sokolov [8], and a dense set of inadmissible domain dimensions has also been disclosed. The Dirichlet problem occurs here when the static or geometric tangential boundary conditions are given in the coordinate line directions. The Dirichlet problem for the system of equilibrium equations has been examined in [7], and a geometric problem with oblique (not coordinate) tangential boundary conditions has been considered in [8] for a one-sheeted hyperboloid. As an experiment realizing the Dirichlet problem, Vlasov demonstrated the model of a thin-walled shell in the shape of a one-sheeted hyperboloid, whose edges were fixed in conformity with the case of one of the coordinate stress resultants in the plane tangent to the shell surface vanishing at each edge. For some dimensions corresponding to incorrectness of the problem the shell would possess high deformability in the case of a special method of loading its lateral surface. This fact shows that lack of a solution of the membrane theory equations corresponds actually to a real change in shell behavior.

In [9] Gol'denveizer has shown that taking account of the everywhere dense set of shell dimensions for which there is no solution to the problem does not always have practical meaning, since within the span of membrane theory we are interested only in solutions whose variability indices are not too great, and the denseness of such solutions should be of a specific order.

Conditions for the existence and correctness of solutions of the static problem are considered herein for shells of revolution of negative curvature for different tangential boundary conditions. The boundary value problem is to determine the solution of a second order equation with a skew derivative given on the boundary.

1. The membrane state of stress of a shell of revolution is described by a system of equilibrium equations [1] which can be written as follows for the coordinate stress resultants T and S in a plane tangent to the shell middle surface

$$\frac{\partial S}{\partial \beta} + \frac{\partial}{\partial z}\frac{rT}{A} = -rX + rr'Z \qquad (A = \sqrt{1 + r'^2}) \tag{1.1}$$
$$r''\frac{\partial}{\partial \beta}\frac{rT}{A} + \frac{1}{r}\frac{\partial r^2 S}{\partial z} = -rAY - rA^2\frac{\partial Z}{\partial \beta}$$

where $r = r(z)$ is the radius in a section perpendicular to the z-axis of revolution of the shell, the prime denotes differentiation with respect to the axial coordinate z, A is the coefficient of the first quadratic form, β is the angular coordinate, T and S are the normal and tangential stress resultants, respectively, in a shell element bounded by coordinate lines, X, Y, Z are the external loading components.

The solution of the system of equations (1.1) in a rectangular domain

$$\Omega\,\{0 \leqslant z \leqslant H,\ 0 \leqslant \beta \leqslant 2\pi\}$$

is considered for boundary conditions of the form

$$k_1 T(0, \beta) + k_2 S(0, \beta) = R_0(\beta), \quad k_1^2 + k_2^2 \neq 0 \tag{1.2}$$
$$k_3 T(H, \beta) + k_4 S(H, \beta) = R_1(\beta), \quad k_3^2 + k_4^2 \neq 0, \quad k_i = \text{const}$$

We call the problem correct if there exists a unique solution for a given H, and the unique solution exists for a sufficiently small change in H.

To investigate the correctness of the problem, let us consider the homogeneous equations corresponding to (1.1), (1.2) since it is necessary that the homogeneous problem have just a trivial solution for the solution to exist in the general case. In the case of the homogeneous problem the system of equations (1.1) can be reduced to one equation by using the stress function Φ

$$\frac{\partial^2 \Phi}{\partial \beta^2} = \frac{1}{rr''}\frac{\partial}{\partial z} r^2 \frac{\partial \Phi}{\partial z}, \qquad T = -\frac{A}{r}\frac{\partial \Phi}{\partial \beta}, \quad S = \frac{\partial \Phi}{\partial z} \tag{1.3}$$

According to (1.3), the homogeneous boundary conditions corresponding to (1.2) can be written as

$$z = 0, -k_1\frac{A}{r}\frac{\partial \Phi}{\partial \beta} + k_2\frac{\partial \Phi}{\partial z} = 0, \quad k_1^2 + k_2^2 \neq 0$$
$$z = H, -k_3\frac{A}{r}\frac{\partial \Phi}{\partial \beta} + k_4\frac{\partial \Phi}{\partial z} = 0, \quad k_3^2 + k_4^2 \neq 0 \tag{1.4}$$

For shells of negative curvature $rr'' > 0$, and hence, (1.3) is of hyperbolic type. We henceforth assume that the function $r(z)$ has the required number of continuous derivatives. The correctness of the problem described by (1.3), (1.4) is defined by a theorem.

Theorem. A set of values k_1, k_2, k_3, k_4 exists for given dimensions of the domain of definition of the solution so that the problem (1.3), (1.4) will be correct.

Proof. Let us represent the solution of (1.3) as a formal series $\Phi = \sum_n \Phi_n(z, \beta)$, where the terms of the series satisfy (1.3) and have the form

$$\Phi_n(z, \beta) = \varphi_1(z, n)\cos n\beta + \varphi_2(z, n)\sin n\beta \tag{1.5}$$

Substituting (1.5) into (1.3) and (1.4), we obtain that $\varphi_1(z)$ and $\varphi_2(z)$ satisfy the equation

$$(r^2\varphi')' + n^2 rr''\varphi = 0 \tag{1.6}$$

and boundary conditions of the form

$$z = 0, \quad -nl_1\varphi_1 + k_2\varphi_2' = 0, \quad nl_1\varphi_2 + k_2\varphi_1' = 0 \qquad (l_1 = -k_1A(0)/r(0))$$
$$z = H, \quad -nl_3\varphi_1 + k_4\varphi_2' = 0, \quad nl_3\varphi_2 + k_4\varphi_1' = 0 \qquad (l_3 = -k_3A(H)/r(H)) \tag{1.7}$$

Let us represent φ_1 and φ_2 as

$$\varphi_1 = A\psi_1 + B\psi_2, \quad \varphi_2 = C\psi_1 + D\psi_2, \quad A, B, C, D = \text{const} \tag{1.8}$$

where ψ_1 and ψ_2 satisfy (1.6) and the initial conditions

$$\psi_1(0) = 0, \quad \psi_1'(0) = n, \qquad \psi_2(0) = 1, \quad \psi_2'(0) = 0. \tag{1.9}$$

Let us substitute (1.8) into (1.7). The system of equations in A, B, C, D obtained here can have a nontrivial solution when

$$\begin{vmatrix} 0 & -l_1 & k_2 & 0 \\ k_2 & 0 & 0 & l_1 \\ -nl_3\psi_1(H) & -nl_3\psi_2(H) & k_4\psi_1'(H) & k_4\psi_2'(H) \\ k_4\psi_1'(H) & k_4\psi_2'(H) & nl_3\psi_1(H) & nl_3\psi_2(H) \end{vmatrix} = 0 \tag{1.10}$$

Condition (1.10) corresponds to possible incorrectness of the problem, and if the determinant is not zero, the problem is solvable. Condition (1.10) is equivalent to the following:

$$nl_1l_3\psi_1(H) + k_2k_4\psi_2'(H) = 0, \qquad nl_3k_2\psi_2(H) - l_1k_4\psi_1'(H) = 0 \tag{1.11}$$

which will be called the incorrectness conditions. Let us make the change of variable

$$\psi = \frac{u(\xi)}{(r^3r'')^{1/4}}, \qquad \xi = \int_0^z \left(\frac{r''}{r}\right)^{1/2} dz$$

Then (1.6) is converted to the form [10]

$$\frac{d^2u}{d\xi^2} + [n^2 - Q(\xi)]\,u = 0, \qquad Q(\xi) = Q(z(\xi)) = \frac{r^2}{(r^3r'')^{3/4}}\frac{d}{dz}r^2\frac{d}{dz}(r^3r'')^{-1/4}$$

For fairly large n its solution can be represented in the asymptotic form

$$u_1 = c_1 \sin n\xi + O(n^{-1}), \qquad u_1' = nc_1 \cos n\xi + O(1)$$
$$u_2 = c_2 \cos n\xi + O(n^{-1}), \qquad u_2' = -nc_2 \sin n\xi + O(1)$$

and putting $c_1 = \sqrt[4]{r^5(0)/r''(0)}$, $c_2 = \sqrt[4]{r^3(0)\,r''(0)}$, respectively, we obtain

$$\psi_1(z) = \frac{c_1 \sin n\xi(z)}{(r^3r'')^{1/4}} + O\left(\frac{1}{n}\right), \qquad \psi_1'(z) = nc_1\left(\frac{r''}{r^5}\right)^{1/4}\cos n\xi(z) + O(1)$$
$$\psi_2(z) = \frac{c_2 \cos n\xi(z)}{(r^3r'')^{1/4}} + O\left(\frac{1}{n}\right), \qquad \psi_2'(z) = -nc_2\left(\frac{r''}{r^5}\right)^{1/4}\sin n\xi(z) + O(1)$$

Substituting this latter into the incorrectness conditions (1.11), we obtain

$$\left[l_1l_3\left(\frac{r^5(0)}{r^3(H)\,r''(0)\,r''(H)}\right)^{1/4} - k_2k_4\left(\frac{r^3(0)\,r''(0)\,r''(H)}{r^5(H)}\right)^{1/4}\right]\sin n\xi(H) + O\left(\frac{1}{n}\right) = 0$$
$$\left[k_2l_3\left(\frac{r^3(0)\,r''(0)}{r^3(H)\,r''(H)}\right)^{1/4} - l_1k_4\left(\frac{r^5(0)\,r''(H)}{r''(0)\,r^5(H)}\right)^{1/4}\right]\cos n\xi(H) + O\left(\frac{1}{n}\right) = 0 \tag{1.12}$$

Assuming l_1, l_3, k_2, k_4 to be nonzero, let us introduce the notation $p = l_1/k_2$, $q = l_3/k_4$, which denote quantities proportional to the slopes of the stress resultants on the shell edges. Then the incorrectness conditions (1.11) will determine some curves in the (p, q) plane for different n, or points of intersection of the corresponding curves. If $\psi_i(H)$, $\psi_i'(H)$, $i = 1,2$ are not zero, then conditions (1.11) can be represented as

$$n\psi_1(H)\,pq = -\psi_2'(H), \qquad n\psi_2(H)\,q = \psi_1'(H)\,p$$

and they jointly define possibly two points of intersection of two branches of a hyperbola with a straight line. Let us examine the case when $\psi_i(H)$ and $\psi_i'(H)$ may be zero.

Let $\psi_1(H) = \psi_2'(H) = 0$. Then the first of Eqs. (1.11) is satisfied identically. Since $\psi_2(H)$ and $\psi_1'(H)$ are not zero, otherwise the solutions $\psi_i(z)$ would be trivial because of the uniqueness of the solution of the Cauchy problem for (1.6), and would not satisfy the customary conditions (1.9), the incorrectness condition has the form of an equation of a straight line

$$n\psi_2(H)\, q = \psi_1'(H)\, p$$

Let $\psi_2(H) = \psi_1'(H) = 0$. In this case the second equation of (1.11) is satisfied identically, and the first has the form of an equation for a hyperbola

$$n\psi_1(H)\, pq = -\psi_2'(H)$$

According to (1.12), as $n \to \infty$ the incorrectness curves are grouped in the neighborhood of the curves $pq\sqrt{r(0)\, r(H)} = \sqrt{r''(0)\, r''(H)}, \quad q\sqrt{r''(0)\, r(H)} = p\sqrt{r''(H)\, r(0)}$

Only a finite number of incorrectness curves exist outside the neighborhood of these latter, and they have the form of straight lines or hyperbolas. Hence, for given H there always exists a point in the (p, q) plane in whose neighborhood there are no points of incorrectness curves because of the continuous dependence of the left sides of (1.11) on H. The theorem is proved.

Let us consider the incorrectness condition (1.11) for the cases $n = 0$ and $n = 1$ by assuming that l_1, l_3, k_2, k_4 are nonzero.

For $n = 0$, Eq. (1.6) has the solution $\varphi = c_1 \int r^{-2} dz + c_2$, and the incorrectness conditions (1.11) are: $k_2 r^{-1}(0) = 0, \; k_4 r^2(H) = 0$, i.e. the problem is correct for k_2, k_4 not zero.

Let us note that the case $n = 0$ corresponds to the solution of the homogeneous equations (1.1) in the form $T \equiv 0, S = S(z)$. The solution $S \equiv 0, T = T(z)$ follows from (1.3) if Φ is assumed independent of z. The case $n = 0$ is henceforth not taken into account.

For $n = 1$, Eq. (1.6) is transformed by the substitution $\varphi = r^{-1} f$ into $f'' = 0$, and the solution for φ will be: $\varphi = (c_1 z + c_2)\, r$. Defining

$$\psi_1 = r(0)\, z r^{-1}(z), \qquad \psi_2 = [r'(0)\, z + r(0)]\, r^{-1}(z)$$

we obtain the incorrectness conditions in the form

$$l_1 l_3 H r(0)\, r(H) + k_2 k_4 \{r'(0)\, r(H) - [H r'(0) + r(0)]\, r'(H)\} = 0$$
$$k_2 l_3 [H r'(0) + r(0)]\, r(H) - l_1 k_4 [r(H) - H r'(H)]\, r(0) = 0$$

Now, let us examine particular cases of the incorrectness conditions (1.11) when some of the coefficients k_i are zero.

The problem is solvable for any H in the case $l_1 = 0$ $(k_1 = 0)$ and hence is correct since in this case conditions (1.11) have the form $\psi_2(H) = \psi_2'(H) = 0$, and are impossible for a solution $\psi_2(z)$ satisfying conditions (1.9). The problem is analogously always solvable when $k_2 = 0$. When either $l_3 = 0$ or $k_4 = 0$ the problem is also solvable since otherwise the determinant of the Wronskian for the solutions $\psi_1(z)$ and $\psi_2(z)$ would be zero from conditions (1.11), which again contradicts conditions (1.9). For shell theory these cases mean that the problem is always solvable if a coordinate stress resultant is given on one edge, and one different from a coordinate stress resultant on the other.

Let us examine the case when two coefficients are zero. In these cases, the solution can be represented in the form of the following formal series:

$$\Phi = \sum_n \varphi(z, n) \cos(n\beta + \beta_0) \tag{1.13}$$

where $\varphi(z)$ satisfies (1.6). Only four cases of the coefficients k_i and l_i being zero have meaning, and according to (1.4) they correspond to the following boundary conditions for φ :

$$k_2 = k_4 = 0, \qquad \varphi(0, n) = \varphi(H, n) = 0 \tag{1.14}$$

$$l_3 = k_2 = 0, \qquad \varphi(0, n) = \varphi'(H, n) = 0 \tag{1.15}$$

$$l_1 = k_4 = 0, \qquad \varphi'(0, n) = \varphi(H, n) = 0 \tag{1.16}$$

$$l_1 = l_3 = 0, \qquad \varphi'(0, n) = \varphi'(H, n) = 0 \tag{1.17}$$

In all these cases the incorrectness conditions (1.14)–(1.17) correspond to cases of values of the dimension of the domain H agreeing with the zeros or extremum points of the solutions of (1.6). According to the oscillation theory, the number of zeros or extremums in any interval of the z-axis tends to infinity as $n \to \infty$. Therefore, in the neighborhood of each dimension there exists an everywhere dense set of shell dimensions for which the problem is not solvable. The considered cases of two coefficients being zero correspond to cases of either normal T or tangential S loading of the shell edges, and in these cases the problem is always incorrect. Let us investigate these cases in more detail.

An expansion of the external loading components (denoting them by the common symbol P) in a series of the form $P = \sum_n p(z, n) \cos(n\beta + \gamma_0)$ corresponds to a solution of (1.1) in the form (1.3).

Let us call the problem N-correct, if it is correct for the first N terms of the series expansion of the loading.

The definition introduced for the correctness of the problem corresponds to the possibility of considering the existence of a solution for separate terms of an expansion with $n \leqslant N$, and we hence introduce such a definition.

We call the shell dimensions for which the problem will not be solvable for some special kind of loading and given boundary conditions the natural shell dimensions corresponding to this loading and boundary conditions.

In the considered case we determine the natural dimensions for a special loading of the form $p(z, n) \cos(n\beta + \gamma_0)$. Then for the boundary conditions (1.14)–(1.17) the natural dimensions of the shell coincide with the zeros or extremum points of the solution of (1.6). By virtue of the congruence (comparison) theorem and the bounds accepted for the function $r(z)$ $(rr'' > 0)$ in each finite interval there exists only a finite number of natural dimensions for each n, and hence, the problem can always be formulated N-correctly. That is, for the chosen number of terms in the expansion of a given loading, a dimension H can always be selected such that the problem will be correct.

By using the congruence theorem for second order equations we can obtain estimates of the natural dimensions in the cases (1.14)–(1.17).

For $n \geqslant 2$ the estimates are

$$\frac{m\pi}{\alpha n}\left(\frac{\theta_1}{q_1}\right)^{1/2} \leqslant H_m(n) \leqslant \frac{m\pi}{\alpha n}\left(\frac{\theta_2}{q_2}\right)^{1/2} \tag{1.18}$$

$$\theta_1 = \min r^2, \qquad q_1 = \max rr'', \qquad \theta_2 = \max r^2, \qquad q_2 = \min rr''$$

Here $\alpha = 1$ in cases (1.14), (1.17), and $\alpha = 2$ in cases (1.15), (1.16).

For cases (1.14)–(1.16), Eq. (1.6) is compared with the equation

$$\theta_i \varphi'' + n^2 q_i \varphi = 0 \qquad (i = 1, 2)$$

and the estimates (1.18) result from the Picone identity, suitably integrated (G. Sansone, "Ordinary Differential Equations"). The case (1.17) reduces to the case (1.14) if we go from (1.6) to the equation

$$\left(\frac{1}{rr''} u'\right) + \frac{n^2}{r^2} u = 0 \quad (u = r^2\varphi', \ u(0) = u(H) = 0)$$

It is assumed here that θ_i, q_i are not zero.

For $n = 1$ there are no natural dimensions in the case (1.14); in the remaining cases the natural dimensions are determined directly by using the solution of (1.6) which has the form $\varphi = (c_1 z + c_2) r^{-1}(z)$.

Estimates obtained by using the congruence theorem depend essentially on the behavior of the function $r(z)$ in a given interval. For example, let us consider the case (1.14). We reduce (1.6) by the change of variable $\varphi = r^{-1} f$ to the form

$$rf'' + (n^2 - 1) r'' f = 0 \qquad (f(0) = f(H) = 0)$$

It then follows from the congruence theorem that

$$\frac{\pi m}{\sqrt{(n^2-1)\max Q(z)}} \leqslant H_m(n) \leqslant \frac{\pi m}{\sqrt{(n^2-1)\min Q(z)}}, \qquad Q(z) = \frac{r''}{r} \qquad (1.19)$$

If $Q(z)$ decreases rapidly as $z \to \infty$, for example, $Q(z) = {}^1/_4 z^{-2} + O(z^{-2})$, then because of the Kneiser theorem [10] only a finite number of natural dimensions exists on the whole z-axis. If $Q(z)$ grows without limit as $z \to \infty$, then the lower estimate drops out.

Moreover, even the existence of natural dimensions is not a sufficient condition for determination of their approximate values by (1.19).

For example if $\min Q(z) = Q(L)$, $\quad L\sqrt{Q(L)} \leqslant m\pi/\sqrt{n^2-1}$ $\quad$ as $L \to \infty$

then it is impossible to determine the upper bound from (1.19).

In order to be able to improve the estimates, which is important in the evaluation of the first natural dimensions for given n and for m commensurable with n, let us consider variational principles of determining the natural dimensions. Let us examine a functional of the form

$$J = \int_0^L r^2\varphi'^2 - n^2 rr''\varphi^2 dz \qquad (1.20)$$

in the class of functions continuously differentiable and satisfying the boundary conditions of any of the cases (1.14)–(1.17) if L replaces H therein. Let us assume that in each case the interval $[0, L]$ does not contain natural dimensions. In all cases, with the exception of (1.7), the functional takes on the minimum value for the solution of (1.6) which will satisfy the appropriate boundary conditions. Since natural dimensions are not contained in the interval $[0, L]$, then $\varphi = 0$, hence $\min J = 0$. Let us consider each case individually.

Case 1. The functional J is defined on a set of functions satisfying the boundary conditions: $\varphi(0) = \varphi(L) = 0$. Since $J = 0$ only for $\varphi = 0$, then we have $J > 0$ for the remaining φ.

Let us introduce a function of two variables $\varphi(z, \xi)$ which is continuously differentiable with respect to z and satisfies the conditions $\varphi(0, \xi) = \varphi(\xi, \xi) = 0$. Then the functional

$$J(\xi)=\int_0^{\xi} r^2(z)\varphi'^2(z,\xi)-n^2 r(z)r''(z)\varphi^2(z,\xi)\,dz$$

will be greater than zero for $\xi < H_1$, where H_1 is the first natural dimension, i.e. the value of L for which a nonzero solution of (1.6) is possible for the considered boundary conditions. The value $\xi = H_1^*$, for which $J(\xi) = 0$, will be an approximate value of the natural dimension. Evidently the approximate value of the natural dimension obtained in this manner will be the upper bound of the exact value. Since the functional (1.20) has no extremum if the interval contains a natural dimension [12], then there always exists a function $\varphi(z,\xi)$ and a value ξ for which $J(\xi) = 0$.

Cases 2 and 3. The functional is considered in a set of functions satisfying the respective conditions: $\varphi(0) = \varphi'(L) = 0$ or $\varphi'(0) = \varphi(L) = 0$. One of the boundary conditions will be natural in each of these cases: either $\varphi'(L) = 0$ or $\varphi'(0) = 0$, and the function can be selected by satisfying just one boundary condition, either $\varphi(0) = 0$ or $\varphi(L) = 0$, respectively. In these cases the determination of the natural dimension occurs just as in Case 1.

Case 4. The functional J is defined in a set of functions satisfying the boundary conditions $\varphi'(0) = \varphi'(L) = 0$. For such boundary conditions the solution of (1.6) is not unique, and the functional will not be sign-definite. Hence, let us contract the class of functions by introducing an additional condition. Integrating (1.6), and taking account of the boundary conditions, we obtain that the solution should satisfy the condition

$$\int_0^L rr''\varphi\,dz = 0$$

Then the functional takes on the extremal value in the class of functions additionally satisfying this condition. Let us show that if φ is not the solution of (1.6), then the functional will be greater than zero. It is sufficient to establish this for small L since the functional equals zero only for the solution of the equation because of the presence of the extremum. By virtue of the Buniakowski inequality we have

$$\int_0^L rr''\varphi^2 dz = -\int_0^L \varphi'\int_0^z rr''\varphi\,dz \leqslant \left[\int_0^L r^2\varphi'^2\,dz \int_0^L\left(\frac{1}{r}\int_0^z rr''\varphi dz\right)^2 dz\right]^{1/2} \leqslant$$

$$\leqslant \left[\int_0^L r^2\varphi'^2\,dz\int_0^L\frac{1}{r^2}\left(\int_0^z rr''\varphi^2 dz\int_0^z rr''dz\right)dz\right]^{1/2} \leqslant$$

$$\leqslant \left[\int_0^L r^2\varphi'^2\,dz\int_0^L rr''\varphi^2 dz\int_0^L\frac{1}{r^2}\int_0^z rr''dz dz\right]^{1/2}$$

from which

$$\int_0^L n^2 rr''\varphi^2 dz \leqslant n^2\int_0^L\frac{1}{r^2}\int_0^z rr''dz dz\int_0^L r^2\varphi'^2\,dz$$

and the positivity of J for fairly small L is evident.

In the considered case both the boundary conditions are natural, and a class of functions not satisfying any boundary conditions can be considered, and a function satisfying the condition

$$\int_0^{\xi} r(z)r''(z)\varphi(z,\xi)\,dz = 0$$

can be defined as the function $\varphi\ (z,\ \xi)$.

The approximate value of the natural dimension is determined from the equation $J\ (\xi) = 0$.

Let us note that the functional (1.20) has a strong extremum in all the cases considered, and hence, the function $\varphi\ (z,\ \xi)$ can be selected simply continuous and piecewise differentiable. In particular, to simplify the calculations it is sometimes convenient to select the function $\varphi\ (z,\ \xi)$ piecewise linear.

In order to find the approximate value of the next natural dimension we can consider the functional in the interval $[H_1, L)$

$$J\,(n,\,L) = \int_{H_1}^{L} r^2\,\varphi'^2 - n^2 r r'' \varphi^2\,dz$$

Here H_1 is the first eigenvalue. The error in the second eigenvalue will depend on the error in the first, and such a method is applicable only to determine a small quantity of natural dimensions for given n. On the other hand, to evaluate large values of the natural dimensions it is more convenient to utilize their asymptotic estimates obtained, say, by using the congruence theorem.

Let us consider examples of the determination of the natural dimensions.

A. Hyperbolic shell of revolution $r = a\sqrt{b^2 + (z - c)^2}/b$. Equation (1.6) will be of the form

$$\{[b^2 + (z - c)^2]\,\varphi'\}' + \frac{n^2 b^2}{b^2 + (z - c)^2}\,\varphi = 0$$

Let us make the change of variable $z = c + b\ \mathrm{tg}\,\alpha$, we obtain

$$d^2\varphi/d\alpha^2 + n^2\varphi = 0$$

whose solutions are $\cos n\alpha$ and $\sin n\alpha$. Let us define $\psi_1\ (z)$ and $\psi_2\ (z)$ according to conditions (1.9), and substituting in the incorrectness relations (1.11), we obtain

$$[\sqrt{b^4 + (a^2 + b^2)\,c^2}\,\sqrt{b^4 + (a^2 + b^2)\,(H - c)^2}\,k_1 k_3 - a^2 b^2 k_2 k_4]\ \sin n\ \mathrm{arctg}\,\frac{bH}{b^2 - c\,(H - c)} = 0 \qquad (1.21)$$

$$[\sqrt{b^4 + (a^2 + b^2)\,(H - c)^2}\,k_2 k_3 - \sqrt{b^4 + (a^2 + b^2)\,c^2}\,k_4 k_1]\cos n\ \mathrm{arc\,tg}\,\frac{bH}{b^2 - c(H - c)} = 0$$

Hence, it is seen that the solution may not exist for all n, i.e. for any loading when the square brackets in the relations are zero. In this case

$$\frac{k_1}{k_2} = \frac{\pm\,ab}{[b^4 + (a^2 + b^2)\,c^2]^{1/2}}\,, \qquad \frac{k_3}{k_4} = \frac{\pm\,ab}{[b^4 + (a^2 + b^2)\,(H - c)^2]^{1/2}}$$

If k_1, k_2 (or k_3, k_4) are proportional to the cosines of the angles formed by the coordinate stress resultants T and S and the direction of the given tangential stress resultant R on the edge $z = 0$ (or $z = H$), then the obtained conditions mean that any dimension H will be natural when the direction of R at each point of the edge is perpendicular to the direction of the generator of the hyperboloid at this point.

Let us examine the case $k_2 = k_4 = 0$, or $k_1 = k_3 = 0$, and let us set $c = H/2$, which corresponds to a symmetric hyperboloid relative to the coordinate axes. Here

$$\sin n\ \mathrm{arctg}\,\frac{4bH}{4b^2 - H^2} = 0$$

is necessary to satisfy the relationships (1.21).

Solving the equation for the natural dimension H, we first obtain

$$H_m\,(n) = 2b\csc\frac{m\pi}{n}\left(-\cos\frac{m\pi}{n} \pm 1\right) \qquad (m = 1,\ 2,\ 3\ \ldots,\ m < n)$$

Secondly, we have $H = 2b$ for odd n. This latter means that $H = 2b$ will be natural in these two cases for all loadings given by an odd function of β.

For the cases $k_2 = k_3 = 0$, or $k_1 = k_4 = 0$ we have

$$\cos n \operatorname{arctg} \frac{4bH}{4b^2 - H^2} = 0, \quad H_m(n) = 2b \csc \frac{(2m+1)\pi}{2n} \left(-\cos \frac{(2m+1)\pi}{2n} \pm 1\right)$$
$$(m = 0, 1, 2, \ldots;\ m < n)$$

where $H = 2b$ will be the natural dimension for all even n.

B. "Power-law" shell $r = A_0 (z + c)^\nu$, $\nu(\nu - 1) > 0$. In this case (1.6) becomes

$$[(z + c)^{2\nu} \varphi']' + n^2 \nu (\nu - 1)(z + c)^{2\nu - 2} \varphi = 0 \tag{1.22}$$

Its solutions exist in two forms. In case $\theta_n^2 = \nu(\nu - 1)(n^2 - 1) - 1/4 > 0$, we have

$$\varphi_1 = (z + c)^{-\nu + 1/2} \cos \ln (z + c)^{\theta_n}, \quad \varphi_2 = (z + c)^{-\nu + 1/2} \sin \ln (z + c)^{\theta_n}$$

where conditions (1.1) have the form

$$\left[k_1 k_3 \frac{H + c}{c^{\nu - 1} A_0^2} A(0) A(H) - k_2 k_4 \nu (\nu - 1)\right] \sin \ln \left(\frac{H + c}{c}\right)^{\theta_n} = 0$$

$$\left(\frac{1}{2} - \nu\right) \left[\frac{k_1 k_4}{c^\nu} \left(\frac{c}{H + c}\right)^{1/2} A(0) + \frac{k_3 k_2}{(H + c)^\nu} \left(\frac{H + c}{c}\right)^{1/2} A(H)\right] \sin \ln \left(\frac{H + c}{c}\right)^{\theta_n} +$$
$$+ \theta_n \left[\frac{k_1 k_4}{c^\nu} \left(\frac{c}{H + c}\right)^{1/2} A(0) - \frac{k_3 k_2}{(H + c)^\nu} \left(\frac{H + c}{c}\right)^{1/2} A(H)\right] \cos \ln \left(\frac{H + c}{c}\right)^{\theta_n} = 0$$

$$A(z) = \sqrt{1 + A_0^2 \nu^2 (z + c)^{2\nu - 2}}$$

Here, as is seen from the second relationship, no natural dimension exists for all n.

Let us consider the case of natural dimensions coincident with the zeros of the solution of (1.6): $k_2 = k_4 = 0$, which corresponds to assigning the coordinate stress resultant T. Here it is necessary that

$$\sin \ln \left(\frac{H + c}{c}\right)^{\theta_n} = 0, \quad \ln \left(\frac{H + c}{c}\right)^{\theta_n} = m\pi \qquad (m = 1, 2 \ldots)$$

We hence obtain for the natural dimensions

$$H_m(n) = c\left(\exp \frac{m\pi}{\theta_n} - 1\right) \tag{1.23}$$

Let us compare the exact values (1.23) with the estimates obtained by using the congruence theorem. To do this, let us utilize estimates in the form (1.19), which are in this case

$$\frac{\pi m (c + L)}{\sqrt{\nu (\nu - 1)(n^2 - 1)}} \geqslant H_m(n) \geqslant \frac{\pi m c}{\sqrt{\nu (\nu - 1)(n^2 - 1)}}$$

For $\nu = 2$, $n = 2$, $m = 1$ we compute the lower bound

$$H_1(2) \geqslant 1/6 \sqrt{6} \pi c \approx 2.5748c$$

The exact value is in this case

$$H_1(2) = \left(\exp \frac{2\pi}{\sqrt{23}} - 1\right) c \approx 2.7848c$$

For the upper bound we consider the minimum of the functional

$$J(\xi) = A_0^2 \int_0^\xi (z + c)^{2\nu} \varphi'^2 (z, \xi) - n^2 \nu (\nu - 1)(z + c)^{2\nu - 2} \varphi^2 (z, \xi)\, dz$$

We define the function $\varphi(z, \xi)$ as follows:

$$\varphi(z, \xi) = \frac{\varphi(z, \xi)}{(z + c)^{\nu - 1}}, \qquad \varphi_0(z, \xi) = \begin{cases} z/\eta, & 0 \leqslant z \leqslant \eta \\ (\xi - z)/(\xi - \eta), & \eta \leqslant z \leqslant \xi \end{cases}$$

Then we obtain

$$J(\xi) = A_0^2\left[-\left(1+\frac{k-\nu}{3}\right)\xi + \frac{c^2}{\eta} + \frac{(\xi+c)^2}{\xi-\eta}\right], \quad k = (\nu-1)\left[(n^2-1)\nu+1\right]$$

The least value of the upper bound is determined from the condition $\partial J/\partial\eta = 0$. Then

$$\eta = \frac{c\xi}{\xi+2c}, \qquad J(\xi) = \left(-\frac{k-\nu}{3}\xi + 4c + \frac{4c^2}{\xi^2}\right)A_0^2$$

The natural dimension is determined from the condition $J(\xi) = 0$, which yields

$$H_1(n) \leqslant 6\,\frac{1+\sqrt{1+(k-\nu)/3}}{k-\nu}\,c$$

In particular, for $\nu = 2$, $n = 2$, $k = 7$ we obtain

$$H_1(2) \leqslant {}^2/_5\,(3+2\sqrt{6})\,c \approx 3.1596\,c$$

2. After the static problem has been solved, the strains are determined by utilizing the governing relationship of shell theory [1], which can be represented in this case as follows:

$$\varepsilon_1 = \left(1-\frac{\sigma r r''}{A^2}\right)\frac{T}{2Eh}, \qquad \varepsilon_2 = \left(\frac{rr''}{A^2}-\sigma\right)\frac{T}{2Eh}, \qquad \omega = \frac{1+\sigma}{Eh}S \quad (2.1)$$

Here E is the elastic modulus, σ Poisson's ratio, $2h$ the shell thickness. The geometric problem of membrane theory for shells of revolution will be to integrate the equations for the displacements [1] of the form

$$\frac{1}{A^2}\left(\frac{\partial Au}{\partial z} + r''\frac{\partial v}{\partial \beta}\right) = \varepsilon_1 + \frac{rr''}{A^2}\varepsilon_2, \qquad \frac{1}{r}\frac{\partial u}{\partial \beta} + \frac{r}{A}\frac{\partial}{\partial z}\frac{v}{r} = \omega \qquad (2.2)$$

under the following boundary conditions: tangential displacements of the form

$$c_1 u(0,\beta) + c_2 v(0,\beta) = U_0(\beta) \qquad (2.3)$$
$$c_3 u(H,\beta) + c_4 v(H,\beta) = U_1(\beta)$$

are given at points of each edge of the shell, where u, v are the displacements in the directions of the coordinate lines z and β, respectively.

The system of equations (2.2) is hyperbolic for $rr'' > 0$. Let us introduce the function Φ by means of the formulas

$$u = -\frac{r^2}{A}\frac{\partial \Phi}{\partial z}, \qquad v = r\frac{\partial \Phi}{\partial \beta}$$

The homogeneous system (2.3) reduces to (1.3), and an investigation of the correctness of the geometric problem is possible in an analogous manner. When H is a natural dimension, and the geometric problem is conjugate to the static problem, i.e. the direction in which the displacement U is given in (2.3), is perpendicular to the direction of the stress resultant R given in (1.2) at each point of the edge, then the geometric problem is solvable only upon compliance with the conditions presented in [1].

Now, let us examine the purely geometric problem when two tangential displacements are given on each edge of the shell. According to (2.1), (2.2) and taking account of the homogeneous equilibrium equations, the following relationship is valid:

$$\int_0^H\int_0^{2\pi}\left[\left(1-2\sigma\frac{rr''}{A^2}+\frac{(rr'')^2}{A^4}\right)\frac{T^2}{2Eh}+\frac{1+\sigma}{Eh}S^2\right]rA\,d\beta\,dz =$$

$$= \int_0^H\int_0^{2\pi}\left[\left(\varepsilon_1+\frac{rr''}{A^2}\varepsilon_2\right)T+\omega S\right]rA\,d\beta\,dz = \int_0^H\int_0^{2\pi}\left[\left(\frac{1}{A^2}\frac{\partial Au}{\partial z}+\frac{r''}{A^2}\frac{\partial v}{\partial \beta}\right)T+\right.$$

$$\left.+\left(\frac{1}{r}\frac{\partial u}{\partial \beta}+\frac{r}{A}\frac{\partial}{\partial z}\frac{v}{r}\right)S\right]rA\,d\beta\,dz = \int_0^{2\pi}(uT+vS)\,r\,\Big|_0^H d\beta$$

This relationship is a particular case of the more general relationship obtained in [1] for arbitary shells.

Assigning two tangential boundary conditions for the displacements on each shell is equivalent to assigninig the displacements u and v. In the case of the homogeneous geometric problem, $u = v = 0$ on each shell edge. Then the integral will be zero on the boundary. Since $-1 < \sigma \leqslant {}^1/_2$, the expression in the paraentheses in the first integral is always positive, and the integral may be zero only when $T = S = 0$. It follows from (2.1) that $\varepsilon_1 = \varepsilon_2 = \omega = 0$ and the homogeneous geometric problem reduces to the solution of the homogeneous equations (2.2). For the boundary conditions $u = v = 0$ on the edges, the homogeneous equations (2.2) have only trivial solutions, and this means that the geometric problem is solvable in this case.

Moreover, it is easy to see that natural dimensions do not exist, for example, in the case of the following tangential boundary conditions: displacements given on one edge, and stress resultants, or a displacement and stress resultant given on the other.

The author is grateful to A. L. Gol'denveizer for formulating the problem, and for assisting in the research.

BIBLIOGRAPHY

1. Gol'denveizer, A. L., Theory of Elastic Thin Shells. Moscow-Leningrad, Gostekhizdat, 1953.
2. Bourgin, D. G. and Duffin, R., The Dirichlet problem for the vibrating string equation. Bull. Amer. Math. Soc., Vol. 45, №12, 1939.
3. John, F., The Dirichlet problem for a hyperbolic equation. Amer. J. Math., Vol. 63, №1, 1941.
4. Sobolev, S. L., Example of a correct boundary value problem for the vibrating string equation with data on the whole boundary. Dokl. Akad. Nauk SSSR, Vol. 109, №4, 1956.
5. Vakhaniia, N. N., On a boundary value problem with data on the whole boundary for a hyperbolic system equivalent to the vibrating string equation. Dokl. Akad. Nauk SSSR, Vol. 116, №6, 1957.
6. Aleksandrian, R. A., Construction of a complete set of solutions of the homogeneous Dirichlet problem for the vibrating string equation. Dokl. Akad. Nauk SSSR, Vol. 162, №2, 1965.
7. Vlasov, V. Z., On the theory of membrane shells of revolution. Izv. Akad. Nauk SSSR, OTN, №5, 1955.
8. Sokolov, A. M., On the problem of the domain of applicability of membrane theory, and analysis of shells of negative curvature. Izv. Akad. Nauk SSSR, OTN, №5, 1955.
9. Gol'denveizer, A. L., Some mathematical problems of linear theory of thin elastic shalls. Usp. Matem. Nauk Vol. 15, №5(95), 1960.
10. Tricomi, F., Differential Equations. Moscow, IIL, 1962.
11. Stepanov, V. V., Differential Equations, Moscow-Leningrad, Gostekhizdat, 1950.
12. Lavrent'ev, M. A. and Liusternik, L. A., Variational Calculus. Moscow-Leningrad, Gostekhizdat, 1938.

Translated by M. D. F.